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Exact solutions of two complementary 1D quantum many-body systems on the half-line

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Abstract

We consider two particular 1D quantum many-body systems with local interactions related to the root system C_N. Both models describe identical particles moving on the half-line with non-trivial boundary conditions at the origin, and they are in many ways complementary to each other. We discuss the Bethe Ansatz solution for the first model where the interaction potentials are delta-functions, and we find that this provides an exact solution not only in the boson case but even for the generalized model where the particles are distinguishable. In the second model the particles have particular momentum dependent interactions, and we find that it is non-trivial and exactly solvable by Bethe Ansatz only in case the particles are fermions. This latter model has a natural physical interpretation as the non-relativistic limit of the massive Thirring model on the half-line. We establish a duality relation between the bosonic delta-interaction model and the fermionic model with local momentum dependent interactions. We also elaborate on the physical interpretation of these models. In our discussion the Yang-Baxter relations and the Reflection equation play a central role.

1 Introduction

Quantum mechanical models with interactions are, in general, very difficult to solve, but there exist a few important cases where exact solutions are available, allowing them to be understood completely. A prominent example is the delta interaction in one dimension which, in the simplest two-particle case, is defined by the Hamiltonian

$$H = -\partial_x^2 + c\,\delta(x) \tag{1}$$

where c is a real coupling constant and $x \in \mathbb{R}$ the relative coordinate of the two particles, $x = x_1 - x_2$. This latter model is popular because it allows for an explicit solution by simple means: since the delta interaction is restricted to $x = 0$, it only manifests itself in the non-trivial boundary conditions for eigenfunctions $\psi(x)$ of H,

$$\begin{aligned} \psi(0^+) &= \psi(-0^+) \\ \psi'(0^+) - \psi'(-0^+) &= c\,\psi(0^+), \end{aligned} \tag{2}$$

and these can be easily accounted for (we write $\psi(\pm 0^+)$ short for the left-and right limits $\lim_{x\downarrow 0}\psi(\pm x)$, and similarly for the derivative ψ'). The natural generalization of this model to an arbitrary number N of identical particles defines a prominent exactly solvable quantum many-body system which, in the boson case, was solved by Lieb and Liniger [1] and, for the general case of distinguishable particles, by Yang [2] in a seminal paper where the Yang-Baxter relations first appeared.

Interactions localized at points have been studied extensively using the mathematical theory of defect indices; see [3] and references therein. From these studies it is well-known that the delta interaction is only one of many possible local interactions, and a general such interaction can be characterized by four real coupling parameters. This can be easily understood as follows: for a 1D Hamiltonian $H = -\partial_x^2 + \hat{v}$ with an interaction $\hat{v}$ localized at $x = 0$ all eigenfunctions $\psi(x)$ should be smooth everywhere except at $x = 0$, and $(H\psi)(x) = -\psi''(x)$ for non-zero x. Requiring H to be self-adjoint leads to the following condition,

$$\int_{|x|>0} \mathrm{d}x \left(\overline{\phi''(x)}\psi(x) - \overline{\phi(x)}\psi''(x) \right) = 0 \tag{3}$$

for arbitrary wave functions ϕ and ψ, or equivalently

$$[\overline{\phi'}\psi - \overline{\phi}\psi']_{x=0^+} = [\overline{\phi'}\psi - \overline{\phi}\psi']_{x=-0^+}. \tag{4}$$

General boundary conditions are of the form

$$\begin{aligned} \psi(0^+) &= u_{11}\psi(-0^+) + u_{12}\psi'(-0^+) \\ \psi(0^+) &= u_{21}\psi(-0^+) + u_{22}\psi'(-0^+) \end{aligned} \tag{5}$$

(and similarly for ϕ, of course) and are thus parameterized by four complex parameters u_{jk} which, when imposing (4), are reduced to two complex, or equivalently, four real parameters. The boundary conditions in Eq. (2) are obviously contained in this class of boundary conditions, but there are others, most prominently

$$\begin{aligned} \psi'(0^+) &= \psi'(-0^+) \\ \psi(0^+) - \psi(-0^+) &= \lambda\psi'(0^+) \end{aligned} \tag{6}$$

which often has been referred to as delta-prime interaction; see e.g. Section I.4 in [3]. Recently it was shown that these latter boundary conditions arise naturally from the Hamiltonian

$$H = -\partial_x^2 + \lambda\partial_x\delta(x)\partial_x \tag{7}$$

where the second term has a physical interpretation as a local interaction depending also on the momentum $\hat{p} = -\mathrm{i}\partial_x$ [4]. The N-body generalization of this model is exactly solvable by Bethe Ansatz in the indistinguishable particle case when the particles are either bosons or fermions [4,5] but, different from the delta interaction case, not in the general case of distinguishable particles [4]. Still, this model is complementary to the model with the delta interactions for at least three different reasons [4]: firstly, for indistinguishable particles, the delta interaction model is known to be interesting only for bosons (since the delta interaction is trivial on fermion wave functions), whereas the $\hat{p}\delta\hat{p}$-interaction is trivial for bosons and non-trivial for fermions. Secondly, while the delta interaction model for bosons can be obtained as the non-relativistic limit of the quantum sine Gordon model, the $\hat{p}\delta\hat{p}$-interaction model for fermions naturally arises as the non-relativistic limit of the massive Thirring model. Thirdly, there exists an interesting weak coupling duality between the fermionic $\hat{p}\delta\hat{p}$-interaction model and the bosonic delta-interaction model.

As is well-know, exactly solvable many-body systems of particles moving on the full real line are naturally associated with the root system A_{N-1}, and they often allow for extensions to other root systems such that the exact solubility is preserved [6]. An early example was given by Gaudin who solved the C_N root system variant of the delta interaction model for bosons [7], while the general case of this model for arbitrary root systems and distinguishable particles was treated by Sutherland [8]. As pointed out by Cherednik [9], models related to the root system C_N describe

interacting particles on the half line, and the exact solubility requires the so-called Reflection equation to be added to the Yang-Baxter relations. The Reflection equation has played a central role in many exactly solvable systems with a boundary; see e.g. [10] and the review [11].

In this paper we consider the Bethe Ansatz solution of the C_N versions of the models discussed above. Similarly as for the A_{N-1} case we find that the delta interaction model is exactly solvable in this way even for distinguishable particles, but for the model with momentum independent interactions we obtain its exact solution only for indistinguishable particles. We also elaborate on the physical interpretation of these models as describing particles on the half-line with non-trivial boundary conditions at the origin.

To be more specific, the models we discuss in the paper are defined by the following Hamiltonians,

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2c_1 \sum_{j<k} [\delta(x_j - x_k) + \delta(x_j + x_k)] + c_2 \sum_{j=1}^{N} \delta(x_j) \tag{8}$$

(delta interactions) and

$$\begin{aligned} H \quad = \quad & -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\lambda_1 \sum_{j<k} [(\partial_{x_j} - \partial_{x_k})\delta(x_j - x_k)(\partial_{x_j} - \partial_{x_k}) + \\ & +(\partial_{x_j} + \partial_{x_k})\delta(x_j + x_k)(\partial_{x_j} + \partial_{x_k})] + 4\lambda_2 \sum_{j=1}^{N} \partial_{x_j}\delta(x_j)\partial_{x_j} \end{aligned} \tag{9}$$

(local momentum dependent interactions). For simplicity we assume all coupling constants positive so that there are no bound states. Mathematically, the model in Eq. (8) is the C_N variant of the model solved by Yang [2], and Eq. (9) defines the C_N variant of the model discussed in [4].

The plan of the rest of this paper is as follows. In Section 2 we consider the C_N delta-interaction model, starting by deriving the boundary conditions and thus turning the Schrödinger equation $H\psi = E\psi$ into a well-defined mathematical problem. We proceed to the Bethe Ansatz solution of this model where the Yang-Baxter relations and the Reflection equation play a central role. We conclude the section by elaborating on the physical interpretation of this model. In Section 3 we discuss the C_N variant of the $p\delta p$-interaction model, in large parts paralleling our discussion for the delta-interaction in Section 2. We also present a duality relation between the fermionic $p\delta p$-interaction model and the bosonic delta-interaction model. Appendix A gives some details on the verification of the Yang-Baxter relations and the Reflection equation. Appendix B contains a few mathematical facts about the Weyl group of C_N, and Appendix C gives some details on the physical interpretation of these models.

2 Delta-interaction

In this section we provide the exact solution of the C_N delta-interaction in the case of distinguishable particles and elaborate on its physical interpretation.

2.1 Boundary conditions

The Hamiltonian (8) defining the C_N delta-interaction model is only formal, and to determine its eigenfunctions we must first convert it into a set of boundary conditions.

For completeness we start by discussing the Hamiltonian H in Eq. (1), which can be regarded also as the one-particle case of the Hamiltonian in Eq. (8), $N = 1$. The first step to find the eigenfunction ψ of H is to note that the equation $H\psi = E\psi$ for all x is equivalent to $-\psi'' = E\psi$

for $x \neq 0$ together with the boundary conditions in Eq. (2). These boundary conditions are obtained by integrating the equation $H\psi = E\psi$ twice: first from $x = -0^+$ to $x > 0$ and then once more from $x = -0^+$ to $x = 0^+$ yields the first condition in Eq. (2), and integrating from $x = -0^+$ to $x = 0^+$ yields the second condition in Eq. (2). Thus in this case there are two regions free of interactions, $x < 0$ and $x > 0$, linked to each other by the boundary condition at $x = 0$.

For general N, the interaction-terms of the Hamiltonian H in Eq. (8) are restricted to $x_j = \pm x_k$ and $x_j = 0$ for $1 \leq j < k \leq N$, and the eigenfunctions ψ of H therefore obey the simple equation

$$\left(\sum_{j=1}^{N} \partial_{x_j}^2 + E\right)\psi(x_1, \ldots, x_N) = 0 \quad \text{for } x_j \neq \pm x_k \text{ and } x_j \neq 0, \tag{10}$$

and for each of the boundaries of the interaction free regions one gets a pair of boundary conditions similarly to the ones for $N = 1$,

$$\begin{aligned} \psi|_{x_j=\pm x_k+0^+} &= \psi|_{x_j=\pm x_k-0^+} \\ (\partial_{x_j} - \partial_{x_k})\psi|_{x_j=\pm x_k+0^+} - (\partial_{x_j} - \partial_{x_k})\psi|_{x_j=\pm x_k-0^+} &= 2c_1\psi|_{x_j=\pm x_k-0^+} \end{aligned} \tag{11a}$$

$$\begin{aligned} \psi|_{x_j=+0^+} &= \psi|_{x_j=-0^+} \\ \partial_{x_j}\psi|_{x_j=0^+} - \partial_{x_j}\psi|_{x_j=-0^+} &= c_2\psi|_{x_j=0^+} \end{aligned} \tag{11b}$$

(these conditions are obtained by a straightforward generalization of the $N = 1$ argument above, using $\partial_{x_j} \pm \partial_{x_k} = 2\partial_{x_j \pm x_k}$).

Obviously there are now many more regions free of interactions. One such region is $0 < x_1 < x_2 < \ldots < x_N$, and all others are obtained from this by permuting the particle labels, $j \to pj$ with $p \in S_N$ (= permutation group), and/or reflecting some of the coordinates, $x_j \to -x_j$. Thus all regions free of interactions can be characterized as follows,

$$0 < \sigma_1 x_{p1} < \sigma_2 x_{p2} < \ldots < \sigma_N x_{pN} < \infty \tag{12}$$

where $\sigma_j = \pm 1$ and $p \in S_N$; we will refer to these regions as *wedges*. It is important to note that they can be labeled by elements Q in the group

$$W_N := (\mathbb{Z}/2\mathbb{Z})^N \rtimes S_N \tag{13}$$

where the first factor corresponds to the reflections while the second factor corresponds to the permutations of the coordinates,

$$x_{Qj} = \sigma_j x_{pj} \quad \text{for } Q = (\sigma_1, \ldots, \sigma_N; p) \in W_N \text{ with } \sigma_j \in \{\pm 1\} \text{ and } p \in S_N. \tag{14}$$

In the sequel we will therefore use the following convenient notation for the wedges,

$$\Delta_Q : \quad 0 < x_{Q1} < x_{Q2} < \ldots < x_{QN} \tag{15}$$

with $Q \in W_N$. It is interesting to note that the group W_N is isomorphic to the Weyl group of the root system C_N; see e.g. [12].

2.2 Bethe Ansatz

Using the boundary conditions deduced in the previous section we now proceed to determine all eigenfunctions of the C_N delta-interaction, starting by recalling the physical motivation of the Bethe Ansatz below. For that we first consider the Hamiltonian H in Eq. (1). In this case there are eigenfunctions $\psi(x) = \exp(\mathrm{i}kx)$ for $x < 0$ which are equal to a particular linear combination of

$\exp(\mathrm{i}kx)$ and $\exp(-\mathrm{i}kx)$ for $x > 0$. This can be interpreted as scattering by the delta interaction $\propto \delta(x)$ where a plane wave is partly transmitted and partly reflected. Regarding H in Eq. (1) as a two particle Hamiltonian with $x = x_1 - x_2$ the relative coordinate and $k = (k_1 - k_2)/2$ the relative momentum, we can interpret this very fact as scattering of a plane wave solution $\exp(\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2)$ into a linear combination of this wave and another one where the particle momenta k_1 and k_2 are exchanged, $\exp(\mathrm{i}k_2 x_1 + \mathrm{i}k_1 x_2)$. This suggests that an eigenfunction ψ of the N-particle Hamiltonian in Eq. (8) which is equal to a plane wave $\exp(\mathrm{i}\sum_{j=1}^N k_j x_j)$ in one wedge Δ_Q (15) will be transformed into a linear combination of plane waves $\exp(\mathrm{i}\sum_{j=0}^N \tilde{k}_j x_j)$ in any other wedge where $\tilde{k}_j = \sigma_j k_{pj}$, with $\sigma_j = \pm 1$ resulting from the interactions $\propto \delta(x_j)$ which can invert momenta, $k_j \to -k_j$, and $p \in S_N$ resulting from the interactions $\propto \delta(x_j - x_\ell)$ which can interchange momenta, $k_j \leftrightarrow k_\ell$.

We thus see that the group in Eq. (13) naturally appears again, $\tilde{k}_j = k_{Pj}$ for some $P \in W_N$, and the discussion above suggests the following Bethe Ansatz for the eigenfunctions of the Hamiltonian H in Eq. (8),

$$\psi(x) = \sum_{P\in W_N} A_P(Q)\, \mathrm{e}^{\mathrm{i}k_P\cdot x_Q} \qquad \text{for } 0 < x_{Q1} < x_{Q2} < \ldots < x_{QN} \tag{16}$$

with $x = (x_1, \ldots, x_N)$ and $k_P \cdot x_Q \equiv \sum_{j=1}^N k_{Pj} x_{Qj}$, for all $Q \in W_N$. The corresponding eigenvalue is obviously $E = \sum_{j=1}^N k_j^2$.

One now has to take into account the boundary conditions in (11a,b). For each $Q \in W_N$, the wedge Δ_Q (15) participates in N boundaries: $x_{Qi} = x_{Q(i+1)}$ for $i = 1, 2, \ldots (N-1)$ and $x_{Q1} = 0$, and for each of these boundaries we will get two conditions. More specifically, the boundary at $x_{Qi} = x_{Q(i+1)}$ is between the wedges Δ_Q and Δ_{QT_i} where $T_i \in W_N$ is the transposition interchanging i and $(i+1)$, and the conditions implied by Eq. (11a) for $j = Qi$ and $k = Q(i+1)$ are

$$\begin{aligned} A_P(Q) + A_{PT_i}(Q) &= A_P(QT_i) + A_{PT_i}(QT_i) \\ \mathrm{i}(k_{Pi} - k_{P(i+1)})[A_{PT_i}(QT_i) - A_P(QT_i) + A_{PT_i}(Q) - A_P(Q)] &= 2c_1[A_P(Q) + A_{PT_i}(Q)]. \end{aligned} \tag{17a}$$

The boundary at $x_{Q1} = 0$ is between the wedges Δ_Q and Δ_{QR_1} with $R_1 \in W_N$ the reflection of the first argument, i.e., $x_{R_1 j} = x_j$ for $j \neq 1$ and $-x_j$ for $j = 1$, and the conditions at $x_{Q1} = 0$ implied by Eq. (11b) for $j = Q1$ are,

$$\begin{aligned} A_P(Q) + A_{PR_1}(Q) &= A_P(QR_1) + A_{PR_1}(QR_1) \\ \mathrm{i}k_{P1}[A_P(Q) - A_{PR_1}(Q) + A_P(QR_1) - A_{PR_1}(QR_1)] &= c_2[A_P(QR_1) + A_{PR_1}(QR_1)]. \end{aligned} \tag{17b}$$

We thus have $2N(2^N N!)^2$ linear, homogeneous equations for the $(2^N N!)^2$ coefficients $A_P(Q)$. The following beautiful argument due to Yang [2] shows that this system of equations has enough non-trivial solutions and, at the same time, gives a recipe to compute all the $A_P(Q)$.

For that it is important to note that W_N plays a third role: defining

$$(\hat{R})_{Q,Q'} = \delta_{Q',QR} \tag{18}$$

one can write

$$A_P(QR) = \sum_{Q'\in W_N} (\hat{R})_{Q,Q'} A_P(Q') = (\hat{R}A_P)(Q) \tag{19}$$

where the first equality is a trivial consequence of the definition, and in the second we interpret $(\hat{R})_{Q,Q'}$ as elements of an $n \times n$ matrix $\hat{R}$ with $n = 2^N N!$ the rank of W_N. These matrices

obviously define a representation $R \to \hat{R}$ of W_N acting on the coefficients $A_P(Q)$. It is worth noting that this is identical with the so called (right) regular representation of W_N.

We can therefore insert $A_{PT_i}(QT_i) = (\hat{T}_i A_{PT_i})(Q)$ in Eq. (17a), and by a simple computation show that these latter equations are equivalent to

$$A_P = Y_i(k_{P(i+1)} - k_{Pi})A_{PT_i} \tag{20}$$

where we have introduced the operator

$$Y_i(u) = \frac{\mathrm{i}u\hat{T}_i + c_1\hat{I}}{\mathrm{i}u - c_1} \tag{21}$$

and interpret A_P as a vector with $2^N N!$ elements $A_P(Q)$. In the same way we can rewrite the conditions in Eq. (17b) using $A_{PR_1}(QR_1) = (\hat{R}_1 A_{PR_1})(Q)$,

$$A_P = Z(2k_{P1})A_{PR_1} \tag{22}$$

with the operator

$$Z(u) = \frac{\mathrm{i}u\hat{R}_1 + c_2\hat{I}}{\mathrm{i}u - c_2}. \tag{23}$$

It is well-known that the group W_N is generated by the reflection R_1 and the transpositions T_i (see e.g. page 21 in [13]). Thus one can use the identities in Eqs. (20), (22) and (19) to calculate recursively all coefficients $A_P(Q)$ from $A_I(I)$ using the operators Z and Y_i above. It is important to note that there is a possible inconsistency arising from the fact that the representation of an element P in W_N as a product of the T_i's and R_1 is not unique. However, any two such representations can be converted into each other by using the defining relations of the group W_N,

$$\begin{aligned} T_iT_i = 1, \qquad T_iT_j = T_jT_i, \qquad \text{for } |i-j|>1 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \end{aligned} \tag{24a}$$

$$\begin{aligned} R_1R_1 = 1, \qquad R_1T_i = T_iR_1, \qquad \text{for } i>1 \\ R_1T_1R_1T_1 = T_1R_1T_1R_1. \end{aligned} \tag{24b}$$

Thus no inconsistency can arise provided that

$$\begin{aligned} A_{PT_iT_i}(Q) = A_P(Q), \qquad A_{PT_iT_j}(Q) = A_{PT_jT_i}(Q), \qquad \text{for } |i-j|>1 \\ A_{PT_iT_{i+1}T_i}(Q) = A_{PT_{i+1}T_iT_{i+1}}(Q) \end{aligned} \tag{25a}$$

$$\begin{aligned} A_{PR_1R_1}(Q) = A_P(Q), \qquad A_{PR_1T_i}(Q) = A_{PT_iR_1}(Q), \qquad \text{for } i>1 \\ A_{PR_1T_1R_1T_1}(Q) = A_{PT_1R_1T_1R_1}(Q) \end{aligned} \tag{25b}$$

for all $P, Q \in W_N$. Using the recurrence relations (20) and (22) one finds that these conditions hold true if and only if the following operator relations are fulfilled,

$$\begin{aligned} Y_i(-u)Y_i(u) = I, \qquad Y_i(u)Y_j(v) = Y_j(v)Y_i(u), \qquad \text{for } |i-j|>1 \\ Y_i(v)Y_{i+1}(u+v)Y_i(u) = Y_{i+1}(u)Y_i(u+v)Y_{i+1}(v) \end{aligned} \tag{26a}$$

$$\begin{aligned} Z(-u)Z(u) = I, \qquad Z(u)Y_i(v) = Y_i(v)Z(u), \qquad \text{for } i>1 \\ Z(2v)Y_1(u+v)Z(2u)Y_1(u-v) = Y_1(u-v)Z(2u)Y_1(u+v)Z(2v) \end{aligned} \tag{26b}$$

for all real u and v. The validity of this system of equations is necessary and sufficient in order for the Bethe Ansatz above to be consistent and the model at hand to be exactly solvable. The first three relations are the so called *Yang-Baxter relations*, and the last one is the *Reflection equation.*

The validity of these relations for arbitrary $\hat{T}_i$ and $\hat{R}_1$ can be checked by straightforward but somewhat tedious computations (of course, the validity of the Yang-Baxter relation in this case is known since a long time [2], and this seems to be the case also for the Reflection equation [8,10], but for completeness we provide the essential steps in the verification in Appendix A.1).

Thus the Bethe Ansatz (16) is consistent even in the general case of distinguishable particles, and we can calculate all coefficients A_P from A_I using the recurrence relation

$$A_P = \mathcal{W}_P(k) A_I \tag{27}$$

where $\mathcal{W}_P(k)$ is a product of the operators $Y_i(k_{P(i+1)} - k_{Pi})$ and $Z(2k_{P1})$ obtained by using repeatedly (20) and (22).

Interesting special cases of this solution are when the particles are indistinguishable, i.e., when the particles are fermions of bosons. In the former case $\hat{T}_i = -I$, and Eq. (21) implies $Y_i(u) = -I$ independent of the coupling constant c_1. This shows that the delta interaction is trivial for fermions. In the boson case we have $\hat{T}_i = +I$, and $Y_i(u)$ is a non-trivial phase. As discussed in more detail below, there are two different boson cases with different physical interpretations, namely $\hat{R}_1 = -I$ and $\hat{R}_1 = +I$.

2.3 Physical interpretation

As is well-known, the C_N delta-interaction model describe interacting particles on the half-line with particular boundary conditions at the origin [9]. However, the general solution of the C_N delta-interaction model without any restrictions includes many more eigenfunctions than any model on the half line, and the relation between these models is therefore not completely obvious. In this section we discuss the relation of these models in more detail. We also give a physical interpretation of the boundary conditions which occur as limits of particular external potentials restricting the particles to the half line.

As discussed in Appendix B, in any irrep of the group W_N the reflections R_j of the particle coordinate x_j are represented either by $\hat{R}_j = +1$ or -1. For simplicity we now discuss in more detail the cases where all $\hat{R}_j$ are the same, either $+1$ or -1, which from a physical point of view are the most interesting cases. As discussed in Appendix B, these irreps of W_N can be rather easily understood since they are related in a simple way to irreps of S_N. Thus we can impose the following restriction on the eigenfunctions ψ of the Hamiltonian in Eq. (8),

$$(\hat{R}_j\psi)(x_1,\ldots,x_j,\ldots,x_N) \equiv \psi(x_1,\ldots,-x_j,\ldots,x_N) = \pm\psi(x_1,\ldots,x_j,\ldots,x_N). \tag{28}$$

With that assumption we can restrict ourselves to $x_j > 0$, and the boundary conditions in Eq. (11a) and Eq. (11b) reduce to

$$\begin{aligned} \psi|_{x_j=x_k+0^+} &= \psi|_{x_j=x_k-0^+} \\ (\partial_{x_j}-\partial_{x_k})\psi|_{x_j=x_k+0^+} - (\partial_{x_j}-\partial_{x_k})\psi|_{x_j=x_k-0^+} &= 2c_1\psi|_{x_j=x_k+0^+}, \end{aligned} \tag{29a}$$

and

$$\left.\begin{array}{rll} 2\partial_{x_j}\psi|_{x_j=0^+} = & c_2\psi|_{x_j=0^+} & \text{for } \hat{R}_j = +1 \\ \psi|_{x_j=0^+} = & 0 & \text{for } \hat{R}_j = -1 \end{array}\right. , \tag{29b}$$

respectively. These are exactly the boundary conditions obtained from the Hamiltonian

$$H_0 = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2c_1 \sum_{j<k} \delta(x_j - x_k) \tag{30}$$

describing particles on the half-line, $x_j > 0$, and the boundary conditions at the origin given in Eq. (29b).

It is also interesting to note that these later boundary conditions are obtained by allowing the particles to move on the full line, $x_j \in \mathbb{R}$, and adding a particular external potential $\sum_j V(x_j)$ to the Hamiltonian in Eq. (30) which effectively constrains the particles to the half line $x_j > 0$. To be specific, these potentials are given by

$$V(x) = \begin{cases} V_0\Theta(-x) + (c_2/2 - \sqrt{V_0})\delta(x) & \text{if } \hat{R}_j = +1 \\ V_0\Theta(-x) & \text{if } \hat{R}_j = -1 \end{cases}, \tag{31}$$

where $\Theta(-x)$ is the Heaviside function (equal to one for $x < 0$ and zero otherwise), and one has to take the strong coupling limit $V_0 \to \infty$: as shown in Appendix C, in this latter limit the eigenfunctions of the Hamiltonian $H_0 + \sum_j V(x_j)$ on the full line, $x_j \in \mathbb{R}$, coincide with the ones of H_0 on the half-line, $x_j > 0$, and the boundary conditions in Eq. (29b).

As already mentioned, the most important cases in applications are the ones we have considered here, i.e., where all the $\hat{R}_j$ are the same. Nevertheless it would be of interest to consider the implications of allowing the $\hat{R}_j$ to take on different values, in effect dividing the particles into two groups distinguished by their interactions with the boundary.

3 Local momentum-dependent interaction

In this section we discuss the model with local momentum dependent interactions defined by the Hamiltonian in Eq. (9). While most of our discussion is in parallel with the one for the delta interaction model in the previous section, we find that the Bethe Ansatz is consistent only for the indistinguishable particle case. We also present a duality relation between fermionic variant of the model here and the bosonic model discussed in the previous section.

3.1 Boundary conditions

We start by considering the $N = 1$ Hamiltonian H in Eq. (7). To obtain the corresponding boundary conditions we first integrate from $x = -0^+$ to $x = 0^+$ which yields the first condition in Eq. (6), and integrating from $x = -0^+$ to $x > 0$ and then once more from $x = -0^+$ to $x = 0^+$ yields the second condition. As in the delta interaction case, the eigenfunctions ψ of H are then determined by these conditions together with the equation $-\psi'' = E\psi$ for $x \neq 0$. We note that the wave functions $\psi(x)$ on which H in Eq. (7) is defined can be discontinuous at $x = 0$, and to make sense of the interactions we have implicitly used a regularization which amounts to replacing $\psi'(0)$ by $[\psi'(0^+) + \psi'(-0^+)]/2$ (this is discussed in more detail in [4])

It is straightforward to generalize this argument to the N-particle case. Similarly as in the delta interaction case one finds that the eigenfunctions ψ of the Hamiltonian in Eq. (9) are determined by Eq. (10) together with the boundary conditions

$$\begin{aligned} (\partial_{x_j} - \partial_{x_k})\psi|_{x_j=\pm x_k+0^+} &= (\partial_{x_j} - \partial_{x_k})\psi|_{x_j=\pm x_k-0^+} \\ \psi|_{x_j=\pm x_k+0^+} - \psi|_{x_j=\pm x_k-0^+} &= 2\lambda_1(\partial_{x_j} - \partial_{x_k})\psi|_{x_j=\pm x_k-0^+} \end{aligned} \tag{32a}$$

$$\begin{aligned} \partial_{x_j}\psi|_{x_j=0^+} &= \partial_{x_j}\psi|_{x_j=-0^+} \\ \psi|_{x_j=0^+} - \psi|_{x_j=-0^+} &= 4\lambda_2\partial_{x_j}\psi|_{x_j=0^+}. \end{aligned} \tag{32b}$$

3.2 Bethe Ansatz

We now discuss the Bethe Ansatz for the eigenfunctions of the Hamiltonian H defined in Eq. (9). Obviously much of what we said for the delta interaction case carries over straightforwardly to the present case. Due to the different boundary conditions in Eqs. (32a,b) Eqs. (17a,b) are changed to

$$\begin{aligned} \mathrm{i}(k_{Pi}-k_{P(i+1)})[A_{PT_i}(QT_i)-A_P(QT_i)] &= \mathrm{i}(k_{Pi}-k_{P(i+1)})[A_P(Q)-A_{PT_i}(Q)] \\ A_P(QT_i)+A_{PT_i}(QT_i)-A_P(Q)-A_{PT_i}(Q) &= 2\lambda_1\mathrm{i}(k_{Pi}-k_{P(i+1)})[A_P(Q)-A_{PT_i}(Q)] \end{aligned} \tag{33a}$$

$$\begin{aligned} \mathrm{i}k_{P1}[A_P(Q)-A_{PR_1}(Q)] &= \mathrm{i}k_{P1}[A_{PR_1}(QR_1)-A_P(QR_1)] \\ A_P(Q)+A_{PR_1}(Q)-A_P(QR_1)-A_{PR_1}(QR_1) &= 4\lambda_2\mathrm{i}k_{P1}[A_P(QR_1)-A_{PR_1}(QR_1)]. \end{aligned} \tag{33b}$$

We now also use Eq. (19) to convert these in to the recurrence relations

$$A_P = Y_i(k_{P_{i+1}}-k_{P_i})A_{PT_i}, \tag{34}$$

and similarly

$$A_P = Z(2k_{P_1})A_{PR_1} \tag{35}$$

where now

$$Y_i(u) = \frac{\mathrm{i}u\hat{I}-1/\lambda_1\hat{T}_i}{\mathrm{i}u-1/\lambda_1} \tag{36}$$

and

$$Z(u) = \frac{\mathrm{i}u\hat{I}-1/\lambda_2\hat{R}_1}{\mathrm{i}u-1/\lambda_2}. \tag{37}$$

As in the delta interaction case these relations allow to recursively compute all coefficients A_P in terms of A_I, and the conditions for the absence of inconsistencies are identical to (25a,b) of the delta-interaction case, leading to the Yang-Baxter relations (26a) and Reflection equation (26b) but now with the operators (36) and (37). In contrast to the delta-interaction case, we find that these consistency relations are valid only if $\hat{T}_i = \pm I$ for all i (see Appendix A.2 for details). We thus conclude that *the Bethe Ansatz is consistent only if the particles are indistinguishable, i.e., A_I is chosen such that either $\hat{T}_i = I$ or $\hat{T}_i = -I$,* and in these two cases we can compute all coefficients A_P from A_I as

$$A_P = \mathcal{W}_P(k)A_I \tag{38}$$

where $\mathcal{W}_P(k)$ is a product of operators $Y_i(k_{P_{i+1}}-k_{P_i})$ and $Z(2k_{P_1})$ in Eqs. (36) and (37) obtained by using repeatedly (34) and (35).

For $\hat{T}_i = +I$ we get from Eq. (36) that $Y_i(u) = I$ independent of λ_1, and we conclude that *the momentum-dependent interaction is trivial for bosons.* However, for $\hat{T}_i = -I$ (fermions) the $Y_i(u)$ are nontrivial phases. There are two different fermions cases, namely $\hat{R}_1 = \pm I$.

3.3 Duality

It is interesting to note that there exists a simple duality relation between the fermionic $\hat{p}\delta\hat{p}$ model considered here and the bosonic C_N delta-interaction model discussed in Section 2. Since the operators $Y_i(u)$ and $Z(u)$ for the latter model is identical with the ones of the fermions $\hat{p}\delta\hat{p}$ model upon the substitution $\lambda_1 \to 1/c_1$ and $\lambda_2 \to 1/c_2$ (compare Eqs. (21) and (23) for $\hat{T}_i = R_1 = +I$ and Eqs. (36) and (37) for $\hat{T}_i = R_1 = -1$), Eqs. (27) and (38) imply that

$$A_P^{\delta}|_{\hat{T}_i=\hat{R}_1=+I} = A_P^{\hat{p}\delta\hat{p}}|_{\hat{T}_i=\hat{R}_1=-I,\lambda_1\to 1/c_1,\lambda_2\to 1/c_2}, \tag{39}$$

where A_P^δ are the coefficients of Section 2.2 and $A_P^{\hat{p}\delta\hat{p}}$ the ones in Section 3.2. This implies that *the bosonic wave functions of the delta model in Section 2.2 and the fermionic wave functions of the $\hat{p}\delta\hat{p}$-model in Section 3.2 are identical when restricted to the fundamental wedge*

$$\Delta_I : \qquad 0 < x_1 < x_2 < \ldots < x_N, \tag{40}$$

provided that the coupling constants of these models are related as follows,

$$\lambda_1 = \frac{1}{c_1} \quad \textit{and} \quad \lambda_2 = \frac{1}{c_2}. \tag{41}$$

This can be seen also more directly: assuming that the eigenfunction ψ of the Hamiltonian in Eq. (8) is bosonic, $\hat{T}_i = \hat{R} = I$, it is enough to determine it in the fundamental wedge. Moreover, the continuity conditions in Eqs. (11a,b) are fulfilled automatically for boson wave functions, whereas the conditions on the derivatives simplify to

$$\begin{aligned} (\partial_{x_j} - \partial_{x_{j+1}} - c_1)\psi|_{x_j = x_k + 0^+} &= 0 \\ (2\partial_{x_j} - c_2)\psi|_{x_j = 0^+} &= 0 \end{aligned} \tag{42}$$

for all x in the fundamental wedge. In a similar manner one finds that the fermionic eigenfunctions of the Hamiltonian in Eq. (9), $\hat{T}_i = \hat{R} = -I$, are determined by the very same conditions in Eq. (42) with $c_{1,2}$ replaced by $1/\lambda_{1,2}$.

This generalizes the duality previously observed in the A_{N-1} case [4, 5] to the C_N case.

3.4 Physical interpretation

As in the delta interaction case, one can restrict the eigenfunctions ψ of the Hamiltonian in Eq. (9) by imposing the conditions in Eq. (28), reducing the boundary conditions in Eqs. (32a,b) to

$$\begin{aligned} (\partial_{x_j} - \partial_{x_k})\psi|_{x_j = x_k + 0^+} &= (\partial_{x_j} - \partial_{x_k})\psi|_{x_j = x_k - 0^+} \\ \psi|_{x_j = x_k + 0^+} - \psi|_{x_j = x_k - 0^+} &= 2\lambda_1(\partial_{x_j} - \partial_{x_k})\psi|_{x_j = x_k + 0^+} \end{aligned} \tag{43a}$$

and

$$\begin{aligned} \partial_{x_j}\psi|_{x_j = 0^+} &= 0 && \text{for } \hat{R}_j = +1 \\ \psi|_{x_j = 0^+} &= 2\lambda_2 \partial_{x_j}\psi|_{x_j = 0^+} && \text{for } \hat{R}_j = -1 \end{aligned} \tag{43b}$$

where $x_j > 0$. This shows that the eigenfunctions of the C_N Hamiltonian in Eq. (9) with the restriction in Eq. (28) are identical to the ones of the A_{N-1} Hamiltonian

$$H_0 = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\lambda_1 \sum_{j<k} (\partial_{x_j} - \partial_{x_k})\delta(x_j - x_k)(\partial_{x_j} - \partial_{x_k}) \tag{44}$$

restricted to the half-line, $x_j > 0$, and the boundary conditions at the origin given in Eq. (43b).

Moreover, as shown in Appendix C.2, the eigenfunctions ψ above restricted to $x_j > 0$ become identical to the ones of the Hamiltonian $H_0 + \sum_j V(x_j)$ on the full real line, $x_j \in \mathbb{R}$, but with an external potential

$$V(x) = \begin{cases} V_0\Theta(-x) + \sqrt{V_0}\partial_x\delta(x)\partial_x & \text{if } \hat{R}_j = +1 \\ V_0\Theta(-x) + 2\lambda_2\partial_x\delta(x)\partial_x & \text{if } \hat{R}_j = -1 \end{cases} \tag{45}$$

in the limit $V_0 \to \infty$.

4 Concluding remark

As discussed in the Introduction, there exists a 4-parameter family of local interactions [3], and the delta- and $\hat{p}\delta\hat{p}$-interactions only correspond to one-parameter subfamilies each. It is therefore natural to ask: What about the other local interactions? Are there other cases leading to exactly solvable models? It is thus interesting to note that there is a simple physical interpretation of the four parameter family of local interactions which seems more natural than the ones given before [3]: in the simplest case they correspond to the following generalization of the Hamiltonians in Eqs. (1) and (7),

$$H = -\partial_x^2 + c\delta(x) + \lambda\partial_x\delta(x)\partial_x + \gamma\partial_x\delta(x) - \overline{\gamma}\delta(x)\partial_x, \tag{46}$$

which obviously is the most general hermitian Hamiltonian with interactions localized in $x = 0$ and containing only derivatives up to second order (higher derivatives than that do not lead to physically acceptable boundary conditions). This Hamiltonian is formally self-adjoint for arbitrary parameters $c, \lambda \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, and it indeed corresponds to the 4-parameter family of local interactions mentioned above [14]. All these models have natural generalizations to the many-body case, but there is only one case where these latter models are known to be exactly solvable even for distinguishable particles by the coordinate Bethe Ansatz: $(c, \lambda, \gamma) = (c, 0, 0)$. It would be interesting to know if there are other exactly solvable cases. We plan to come back to this question elsewhere [14]. We only mention here that the many-body generalization of the Hamiltonian in Eq. (46) describes identical particles only if $\gamma = 0$, and to find exactly solvable case for non-zero γ therefore requires an extension of Yang's method of solution [2] (which only works for identical particle models).

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Appendix A. Verification of consistency relations

In this appendix we sketch the verification of the consistency relations in Eqs. (26a,b) (Yang-Baxter relations and the Reflection equation).

A.1 Delta-interaction

We start by writing the operators Y_i in the following way:

$$Y_i(u) = a(u) + b(u)\hat{T}_i \tag{A1}$$

where

$$a(u) = \frac{c_1}{\mathrm{i}u - c_1}, \qquad b(u) = \frac{\mathrm{i}u}{\mathrm{i}u - c_1}. \tag{A2}$$

Inserting this expression into the equations in (26a) results in a number of relations between the coefficients $a(u)$ and $b(u)$, one for each equation and different permutation operator. Most of them are trivially fulfilled, but the following ones are non-trivial:

$$\begin{aligned} a(-u)a(u) + b(-u)b(u) &= 1 \\ a(-u)b(u) + b(-u)a(u) &= 0 \end{aligned} \tag{A3}$$

and

$$b(v)a(u+v)a(u) + a(v)a(u+v)b(u) = a(u)b(u+v)a(v). \tag{A4}$$

Inserting $a(u)$ and $b(u)$ from Eq. (A2) they can be verified by straightforward calculations. To verify Eq. (26b) we write the operator Z as

$$Z(u) = \tilde{a}(u) + \tilde{b}(u)\hat{R}_1 \tag{A5}$$

where

$$\tilde{a}(u) = \frac{c_2}{\mathrm{i}u - c_2}, \qquad \tilde{b}(u) = \frac{\mathrm{i}u}{\mathrm{i}u - c_2}. \tag{A6}$$

Substituting this and Eq. (A1) leads to the following non-trivial relation,

$$\begin{aligned} &\tilde{b}(2v)b(u+v)\tilde{a}(2u)a(u-v) + \tilde{b}(2v)a(u+v)\tilde{a}(2u)b(u-v) + \\ &+ \tilde{a}(2v)a(u+v)\tilde{b}(2u)b(u-v) = a(u-v)\tilde{b}(2u)b(u+v)\tilde{a}(2v) \end{aligned} \tag{A7}$$

in addition to

$$\begin{aligned} \tilde{a}(-u)\tilde{a}(u) + \tilde{b}(-u)\tilde{b}(u) = 1 \\ \tilde{a}(-u)\tilde{b}(u) + \tilde{b}(-u)\tilde{a}(u) = 0, \end{aligned} \tag{A8}$$

the validity of which follow from straightforward calculations.

We conclude that the Bethe Ansatz is consistent even for distinguishable particles.

A.2 Local momentum-dependent interaction

In this case we get $Y_i(u)$ as in Eq. (A1) but with

$$a(u) = \frac{\mathrm{i}u}{\mathrm{i}u - 1/\lambda_1}, \qquad b(u) = \frac{-1/\lambda_1}{\mathrm{i}u - \lambda_1}. \tag{A9}$$

With that the two equations in (A3) hold true but the equation in (A4) does not. We therefore conclude that *the Bethe Ansatz is not consistent for distinguishable particles.*

For indistinguishable particles we have $\hat{T}_i = \pm I$ and the Yang-Baxter relations in Eq. (26a) are trivially fulfilled. Moreover, in this case it is also easy to check that the relations (26b) hold true for $\hat{R}_1 = \pm I$.

We conclude that the Bethe Ansatz is consistent in the indistinguishable particle case but not in general.

Appendix B. Representations of the group W_N

In this appendix we discuss the irreducible representations of the group $W_N \equiv (\mathbb{Z}/2\mathbb{Z})^N \rtimes S_N$. In particular we will show the following.

Fact: *There exists a set of irreducible representations of W_N isomorphic to the irreducible representations $\chi_\pm \otimes \rho$, where $\chi_\pm$ is a character (irreducible representation) of the (normal) abelian subgroup $(\mathbb{Z}/2\mathbb{Z})^N$ such that $\chi_\pm(R_j) = \pm 1$ for all $j = 1, 2, \ldots, N$ (same sign for all j) and ρ is an arbitrary irreducible representation of the permutation group S_N.*

To show this we will use the notion of induced representations, following Section 8.2 of [15]. We start by determining the group of characters $X = \mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^N, \mathbb{C})$ of the subgroup $(\mathbb{Z}/2\mathbb{Z})^N$. The fact that it is generated by the reflections R_j obeying the relations (see e.g. page 21 in [13])

$$R_j^2 = I, \qquad j = 1, 2, \ldots, N \tag{B1}$$

implies that the characters $\chi \in X$ are functions such that

$$\chi(R_j) = e^{in_j\pi}, \qquad n_j \in \mathbb{Z} \tag{B2}$$

for all $j = 1, 2, \ldots, N$. The group W_N acts on these characters by

$$(w\chi)(R) = \chi(w^{-1}Rw), \qquad \forall w \in W_N, \chi \in X, R \in (\mathbb{Z}/2\mathbb{Z})^N. \tag{B3}$$

We now determine the orbits of the action of S_N in X, represented by a set χ_i where $i \in X/S_N$. Using the fact that the adjoint action of S_N permutes the reflections R_j, $T_{jk}R_jT_{jk} = R_k$ with T_{jk} the transposition interchanging j and k, we conclude that the orbits of S_N in X can be represented by the characters

$$\chi_k(R_j) = \begin{cases} 1, & j > k \\ -1, & j \leq k \end{cases} \tag{B4}$$

where $j, k = 1, 2, \ldots, N$. For each i let $(S_N)_i$ be that subgroup of S_N consisting of all $P \in S_N$ such that $P\chi_i = \chi_i$, and let further $\tilde{W}_i = (\mathbb{Z}/2\mathbb{Z})^N \cdot (S_N)_i$. The structure of χ_i implies that $(S_N)_i = S_i \times S_{N-i}$. The character χ_i can be extended to all of $\tilde{W}_i$ by setting

$$\chi_i(RP) = \chi(R), \qquad R \in (\mathbb{Z}/2\mathbb{Z})^N, P \in (S_N)_i. \tag{B5}$$

Now let ρ_i be an irreducible representation of $(S_N)_i$ and combine it with the canonical projection $\tilde{W}_i \to (S_N)_i$ to yield an irreducible representation $\tilde{\rho}_i$ of $\tilde{W}_i$. By taking the tensor product of χ_i and $\tilde{\rho}_i$ we can now construct a set of irreducible representations $\chi_i \otimes \rho_i$ of $\tilde{W}_i$. We denote the corresponding induced representation of the whole of W_N by θ_{i,ρ_i}. It follows from Proposition 25 in [15] that all irreducible representations of W_N are isomorphic to such a representation θ_{i,ρ_i}. In particular setting $i = 0$ and $i = N$ we arrive at the claim stated in the Fact at the beginning of the section.

Appendix C. Physical interpretation of boundary conditions

In this appendix we substantiate the physical interpretation of the boundary conditions of the C_N models given in Sections 2 and 3 in the main text.

C.1 Delta-interaction

We first recall the eigenfunctions ψ of the one particle Hamiltonian in Eq. (1). Since this Hamiltonian is invariant under the reflection $x \to -x$ these eigenfunctions can be chosen such that $\psi(x) = \pm\psi(-x) \equiv \psi_\pm(x)$, and they can be computed using the Ansatz

$$\psi_\pm(x) = \begin{cases} \mathrm{e}^{-\mathrm{i}kx} + A_\pm \mathrm{e}^{\mathrm{i}kx} & \text{for } x > 0 \\ \pm\left(\mathrm{e}^{\mathrm{i}kx} + A_\pm \mathrm{e}^{-\mathrm{i}kx}\right) & \text{for } x < 0 \end{cases}, \tag{C1}$$

and the boundary conditions in Eq. (2) determine the constants $A_\pm$ as follows,

$$A_+ = \frac{\mathrm{i}k + c/2}{\mathrm{i}k - c/2}, \qquad A_- = -1 \tag{C2}$$

with A_- being independent of c corresponding to the fact that the delta interaction is trivial (i.e. invisible) for fermions. Obviously, these eigenfunctions obey

$$-\psi_+''(x) = k^2\psi_+(x) \quad \text{for } x>0 \text{ and } \quad \psi'(0^+) = (c/2)\psi(0^+) \tag{C3}$$

and

$$-\psi_-''(x) = k^2\psi_-(x) \quad \text{for } x>0 \text{ and } \quad \psi_-(0^+) = 0, \tag{C4}$$

which is the simplest non-trivial case $N=1$ of the general relation between the C_N model and the A_{N-1} model discussed in Section 4.1.

We now show that these eigenfunctions $\psi_\pm(x)$ for $x>0$ are identical to the ones of the Hamiltonians

$$H_\pm = -\partial_x^2 + V_0\Theta(-x) + g_\pm\delta(x) \tag{C5}$$

with

$$g_+ = c/2 - \sqrt{V_0} \quad \text{and} \quad g_- = 0 \tag{C6}$$

in the limit $V_0 \to \infty$. To show this we determine the eigenfunctions $\phi_\pm$ of $H_\pm$ with the Ansatz

$$\phi_\pm = \begin{cases} \mathrm{e}^{-\mathrm{i}kx} + B_\pm \mathrm{e}^{\mathrm{i}kx}, & \text{for } x>0 \\ C_\pm e^{\omega x}, & \text{for } x<0 \end{cases}, \tag{C7}$$

and by straightforward computations we find

$$B_\pm = \frac{\mathrm{i}k + (\omega + g_\pm)}{\mathrm{i}k - (\omega + g_\pm)} \quad \text{and} \quad \omega = \sqrt{V_0 - k^2} \tag{C8}$$

for $V_0 > k^2$. We thus see that

$$A_\pm = \lim_{V_0\to\infty} B_\pm \tag{C9}$$

provided that $g_\pm$ are chosen as in Eq. (C6). This shows that the eigenfunctions ϕ_+ of the Hamiltonian H_+ on the full line in the limit $V_0 \to \infty$ become equal to $\psi_+(x)$ for $x>0$ (and zero otherwise), and similarly for ϕ_-, ψ_- and H_-.

This computation substantiates the physical interpretation of the C_N model in case $N=1$. However, since this interpretation only involves the boundary conditions at $x_j = 0$ which are not affected by the inter-particle interactions, this argument immediately generalizes to the $N>1$ particle case.

C.2 Local momentum dependent interaction

The discussion for the Hamiltonian in Eq. (7) is completely analogous to the one for the Hamiltonian in Eq. (1) given above, and we therefore only write down the formulas which change.

Eq. (C1) determining the even and odd eigenfunctions $\psi_\pm$ remains the same but A_+ and A_- are (essentially) interchanged,

$$A_+ = 1, \quad A_- = \frac{\mathrm{i}k + 1/2\lambda}{\mathrm{i}k - 1/2\lambda}, \tag{C10}$$

where now the boson eigenfunction is unaffected by the interaction. Moreover, these eigenfunctions solve the following problems on the half axis,

$$-\psi_+''(x) = k^2\psi_+(x) \quad \text{for } x>0 \text{ and } \quad \psi_+'(0^+) = 0 \tag{C11}$$

and

$$-\psi_-''(x) = k^2\psi_-(x) \quad \text{for } x>0 \text{ and } \quad \psi_-'(0^+) = 2\lambda\psi_-(0^+). \tag{C12}$$

The physical interpretation of these boundary conditions is provided by the following Hamiltonians with external fields,

$$H_{\pm} = -\partial_x^2 + V_0\Theta(-x) + \tilde{g}_{\pm}\partial_x\delta(x)\partial_x \tag{C13}$$

which has eigenfunctions as in Eq. (C7) but with

$$B_{\pm} = \frac{\mathrm{i}k + \omega/(1+\omega\tilde{g}_{\pm})}{\mathrm{i}k - \omega/(1+\omega\tilde{g}_{\pm})} \quad \text{and} \quad \omega = \sqrt{V_0 - k^2}, \tag{C14}$$

which converge to $A_{\pm}$ for $V_0 \to \infty$ provided that, for example,

$$\tilde{g}_{+} = \sqrt{V_0} \quad \text{and} \quad \tilde{g}_{-} = 2\lambda. \tag{C15}$$

References

[1] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, Phys. Rev. **130**, 1605 (1963)

[2] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. **19**, 1312 (1967)

[3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, Heidelberg (1988)

[4] H. Grosse, E. Langmann and C. Paufler, Exact solution of a 1D many-body system with momentum dependent interactions, J. Phys. A: Math. Gen. **37**, 4579 (2004); corrected version on arXiv:math-ph/0401003

[5] T. Cheon and T. Shigehara, Fermion-boson duality of one-dimensional quantum particles with generalized contact interactions, Phys. Rev. Lett. **82**, 2536 (1999)

[6] M. A. Olshanetsky and A. M. Perelomov, Quantum completely integrable systems connected with semisimple Lie algebras, Lett. Math. Phys. **2** (1977) 7

[7] M. Gaudin, Boundary energy of a bose gas in one dimension, Phys. Rev. A **4** (1971) 386

[8] B. Sutherland, Nondiffractive scattering from kaleidoscopes, J. Math. Phys. **21** (1979) 1770

[9] I. V. Cherednik, Factorizing particles on a half line and root systems, Theor. Math. Phys. **61** (1984) 977 [Teor. Mat. Fiz. **61** (1984) 35]

[10] E. K. Sklyanin, Boundary conditions for integrable quantum systems, J. Phys. A **21** (1988) 2375

[11] P. P. Kulish, Yang-Baxter equation and reflection equations in integrable models, in: H. Grosse and L. Pittner (eds.), *Low-dimensional models in statistical physics and quantum field theory*, Lecture Notes in Physics, Springer-Verlag (1996) 125–144 [arXiv:hep-th/9507070]

[12] M. A. Olshanetsky and A. M. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rept. **94** (1983) 313

[13] M. Geck and G. Pfeiffer, *Characters of Coxeter Groups and Iwahori-Hecke Algebras*, Oxford science publications, Oxford (2000)

[14] M. Hallnäs, E. Langmann and C. Paufler (in preparation)

[15] J. -P. Serre, *Linear representations of finite groups*, Springer-Verlag, New York (1977)